

Strong Hyperbolicity

*VII Mexican School on Gravitation
and Mathematical Physics:
Relativistic Astrophysics and Numerical Relativity*

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Outline:

- Historical Remarks
- Cauchy problem, characteristics
- Cauchy-Kowalevski and Holmgren theorems
- Learning from Fourier Transform
- Strong hyperbolicity
- Weak hyperbolicity and lower order terms
- Symmetric hyperbolicity
- Applications
 - ADM-BSSN hyperbolicity
 - Subsidiary system hyperbolicity
 - Boundary values (Olivier)

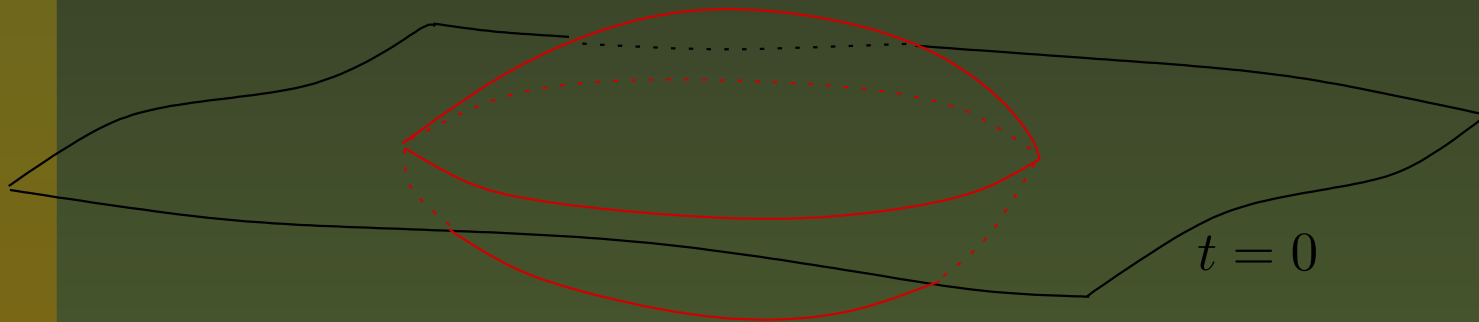
Historical Remarks:

- The theory of PDE's is new. Was developed at around the 50's.
- Much latter than the physical theories were stated (Fluids, Elasticity, Maxwell, QM, GR, YM)
- When people attempted to compute numerical solutions.
- Is a contemporary and very active field of research. In particular linked to numerical simulations.

Cauchy problem

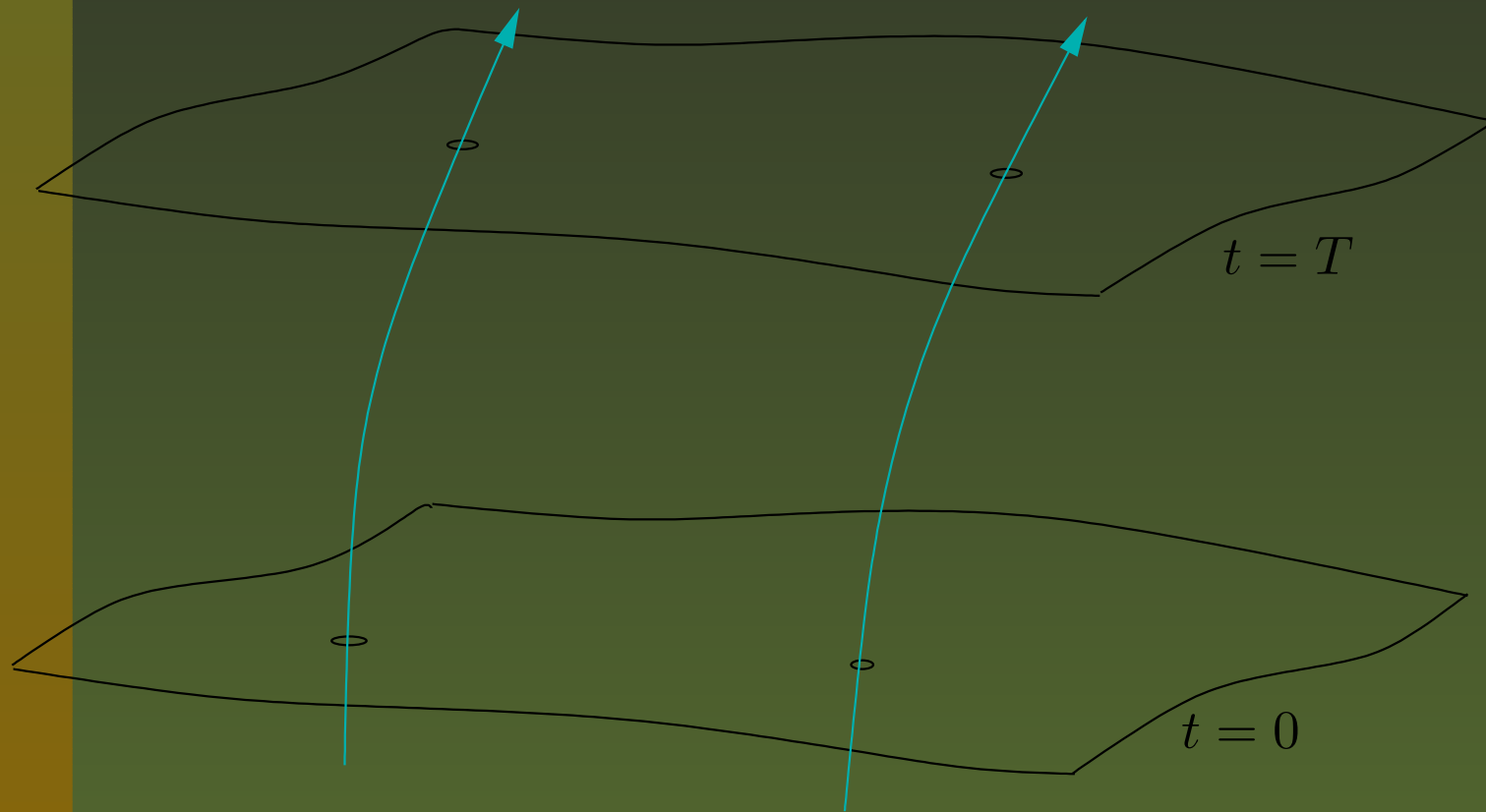
Consider a system of first order PDE's

$$A^\mu(x)\partial_\mu u = F(x) \quad u \text{ is a vector, } A^\mu \text{ a matrix}$$



Cauchy problem

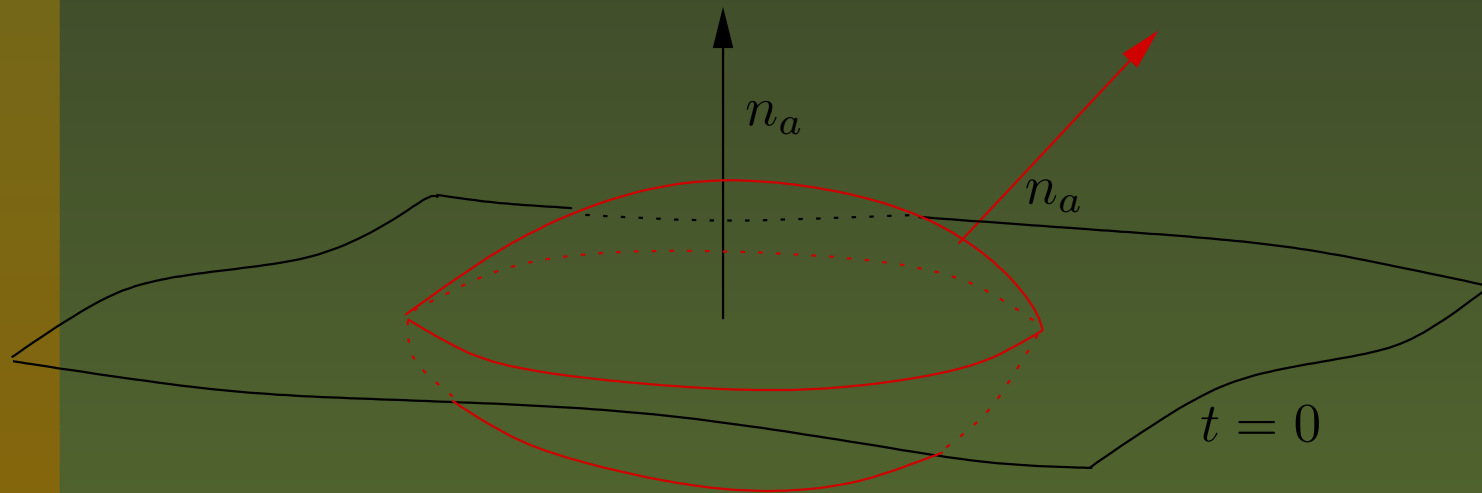
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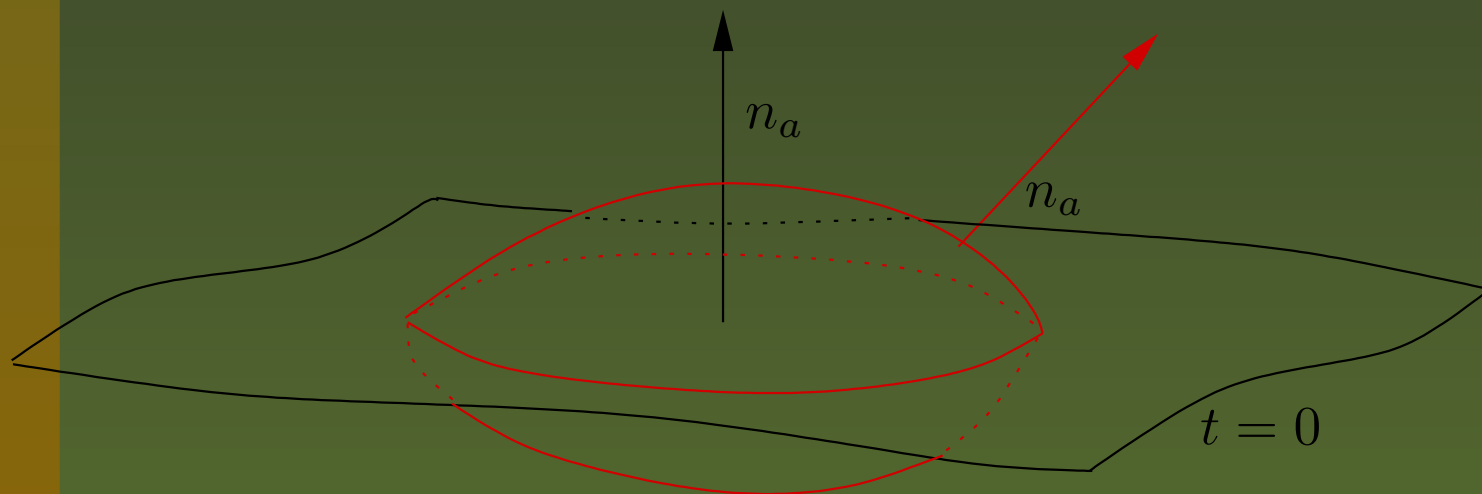
- If $A^\mu n_\mu$ is invertible we can solve for the field off the surface if we know the field on the surface.
- In such a case the surface is called non-characteristic or Cauchy

$$A^\mu(x)\partial_\mu u = F(x) \quad u \text{ is a vector, } A^\mu \text{ a matrix}$$



Cauchy problem

- Cauchy-Kowalevski Theorem: If $A(x)$ and $F(x)$ are analytic then for every analytic initial data $f(\hat{x})$ there exists a region around Σ_0 such that a solution to the system exists this solution is analytic and furthermore $u|_{\Sigma_0} = f(\hat{x})$.
- Holmgren's Theorem: even when f is not analytic, if a solution exists, it is unique in a domain bounded by characteristics surfaces.
- Everything seemed to be O.K.



Cauchy problem: examples

$$\phi_{,11} + \phi_{,22} = f$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{,1} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{,2} = \begin{pmatrix} f \\ 0 \end{pmatrix} \quad u := \phi_{,1} \quad v := \phi_{,2}$$

- In this case $0 = \det(A^\mu n_\mu) = n_1^2 + n_2^2 \implies n_\mu = 0$ so there is no characteristic surfaces.
- Elliptic equations, if you know the solution on a nbh. of a point you know it everywhere.
- **Something funny...**

Cauchy problem: examples

$$\phi_{,11} - \phi_{,22} = f$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{,1} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{,2} = \begin{pmatrix} f \\ 0 \end{pmatrix} \quad u := \phi_{,1} \quad v := \phi_{,2}$$

- In this case $0 = \det(A^\mu n_\mu) = n_1^2 - n_2^2 \implies n_\mu = (1, \pm 1)$ so there are characteristic surfaces.
- Hyperbolic equations, characteristics are maximum propagation speeds.
- Concept of domain of dependence.

Cauchy problem: examples

$$\phi_{,11} + \phi_{,2} = f$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{,1} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{,2} = \begin{pmatrix} f + v \\ 0 \end{pmatrix} \quad u := \phi_{,1} \quad v := \phi_{,2}$$

- In this case $0 = \det(A^\mu n_\mu) = n_1^2 \implies n_\mu = (0, 1)$ so there is only a characteristic surface.
- Parabolic equations, if you know the solution on a nbh. of a point you know it everywhere.
- **Something funny...**

Constant Coefficient Systems

$$u_t(t, x) = P(\partial)u(t, x) \quad P(\partial_j) = \sum_{|\alpha| \leq m} A^\alpha D_\alpha \quad \text{non-characteristic surface}$$

$$u(t, x) = \sum_k e^{ix \cdot k} \hat{u}(t, k) \quad \hat{u}(t, k) := \int e^{-ix \cdot k} u(t, x)$$

$$\hat{u}_t = P(ik)\hat{u} \quad \Longrightarrow \quad \hat{u}(t, k) = e^{P(ik)t} u(0, k)$$

$$f(x) = \sum_{|k| < r} e^{ix \cdot k} \hat{f}(k) \quad \Longrightarrow \quad u(t, x) = \sum_{|k| < r} e^{ix \cdot k} e^{P(ik)t} u(0, k) \hat{f}(k)$$

- Good for finite series, essentially Cauchy-Kowalevski theorem.

Constant Coefficients: Stability

- When is the solution at a given time has a continuous dependence with respect to initial data? **Hadamard**

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- **For Laplacian** it is not bounded, grows exponentially with $|k|$.

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- **For Wave** it is bounded by one, $P(ik)$ is diagonalizable and has imaginary eigenvalues.

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- **For Heat** it is not, either $|e^{P(ik)}|$ or $|e^{-P(ik)}|$ grows exponentially with $|k|^2$.

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- **Without stability you can not compute anything!**

Strong Hyperbolicity

- Restrict to first order systems $P(\partial) = A^a \partial_a + B$
 - Most systems can be put in first order
 - More on this latter for second order systems

Strong Hyperbolicity

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- Definition of Strong hyperbolicity for constant coefficient systems

Strong Hyperbolicity

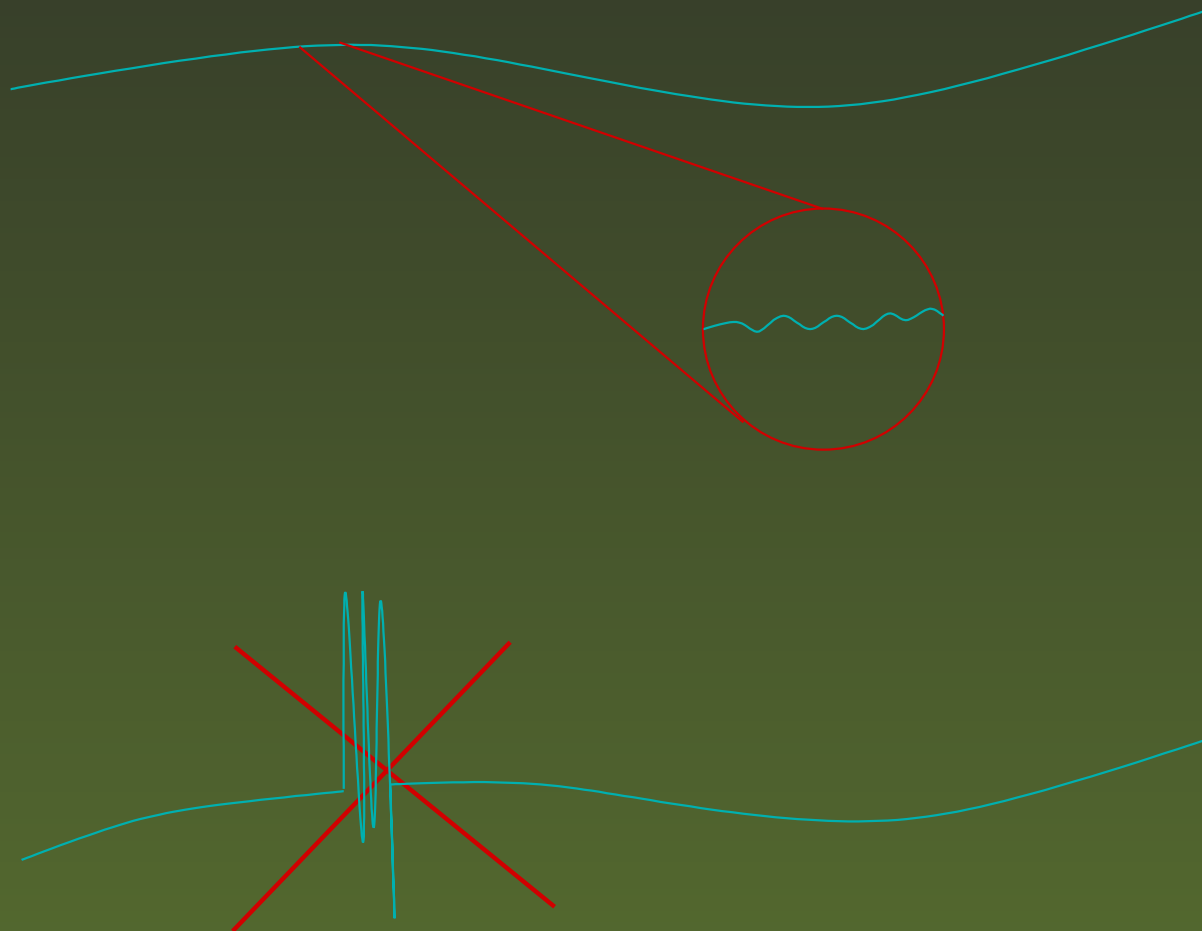
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Strong Hyperbolicity

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- Definition of Strong hyperbolicity for constant coefficient systems
- Alternatively: It is SH if there exist $H = H(k)$ positive definite bilinear form such that $H(k)A^a k_a$ is symmetric.
- Symmetric hyperbolicity: If H does not depend on k_a . Most physical systems can be cast in Symmetric Hyp. form.

Why does it works?

- In the strongly hyperbolic case this limit does not depends on lower order terms



The Symmetric Hyperbolic World

- Geroch Cargese Lectures

The Symmetric Hyperbolic World

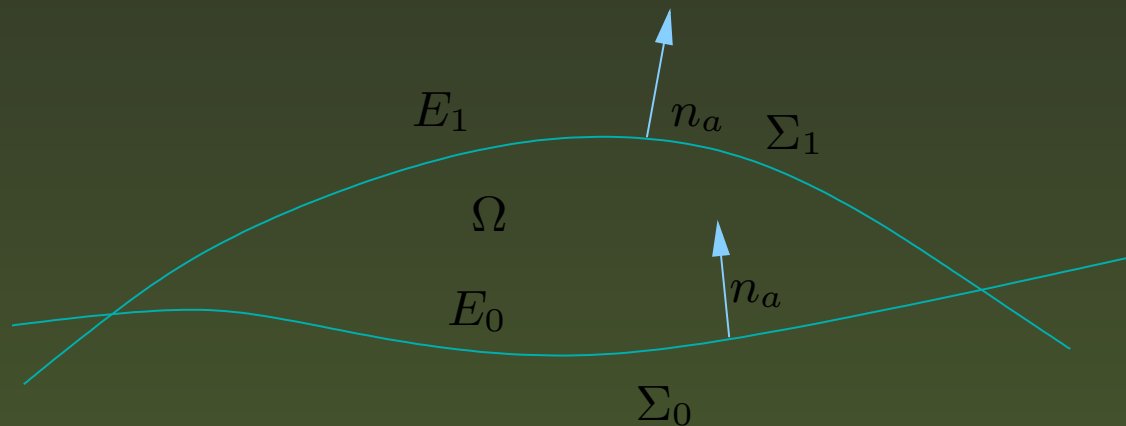
- Different times (If the system is SH for one choice of time, then it is so in an open nbh. of time choices).

The Symmetric Hyperbolic World

- Localization (If the data vanishes in a region, then it vanishes in a whole region which extends up to the characteristic surfaces).

The Symmetric Hyperbolic World

$$J^a := u(Ht^a + HA^a)u \quad \partial_a J^a = u(t^a \partial_a H + \partial_a A^a + HB)u$$



$$E_1 = E_0 + \int \partial_a J^a \quad E := \int J^a n_a$$

If we deform n_a , $J^a n_a > 0$ until $\text{Kern}(J^a n_a) \neq \emptyset$ that is we encounter a characteristic surface.

The Symmetric Hyperbolic World

- Uniqueness
- Finite propagation speed
- Localization
- Existence
- Stability
- Can generalize to variable coefficients
- Can generalize to quasilinear systems

The strongly Hyperbolic World

- Many times??? Difficult, but OK
- Finite prop. and Localization??? Indirect (using Holmgren's) but OK
- Existence??? OK
- Stability??? OK
- Generalize to variable coeffs.??? Need pseudo-differential calculus and certain smoothness of $H(x, k)$.
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- Why bother then!

Why bother? Example

Maxwell equations:

$$W_{ij} := \partial_i A_j$$

$$\begin{aligned}\partial_t E_i &= \partial^j W_{ji} - \partial^j W_{ji} - \alpha(\partial^j W_{ij} - \partial_i W_j^j) \\ \partial_t W_{ij} &= \partial_i E_j - \frac{1}{2}\beta e_{ij} \partial^k E_k\end{aligned}$$

This system is symmetric hyperbolic for $\alpha < 0$ and $\beta < -\frac{2}{3}$ (most general symmetrizer built out of the 3-metric). But strongly hyperbolic for all (α, β) such that $\alpha\beta > 0$. Standard Maxwell $\alpha = \beta = 0$

Why bother?

- Can treat second order systems
- Can understand constraint propagation
- Can obtain most general boundary value conditions (Kreiss-Winicour)

First Order Pseudo-Differential Systems

$$\partial_t u = P(u, x, t, D)u := \int p(u, x, t, \omega_i) e^{i\omega_i x^i} \hat{u} d\Omega$$

- The above system is said to be pseudo-differential of first order if the following limit exists,

$$\lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} p(u, x, t, \lambda \omega_i) := p_1(u, x, t, \omega_i)$$

and does not identically vanishes.

- If furthermore $ip_1(u, x, t, \omega_i)$ has only real eigenvalues and a complete set of eigen-vectors we say the systems is **strongly hyperbolic**.
- Strongly hyperbolic pseudo-differential operators **plus technical smoothness condition** are well posed. [Taylor, Kreiss-Ortiz-R]
- True, for instance, if the eigenvector multiplicity is constant for each eigenvalue.

Summary

- Strongly hyperbolic differential (and pseudo-differential) systems are well posed \leftrightarrow stability
- There are global energy norms (pseudo-differential operators) \leftrightarrow Connection between truncation error and solution error
- Strongly hyperbolic differential (and pseudo-differential) systems have finite propagation speeds. With domain of dependence given by their characteristic fields \leftrightarrow finite stencils, CFL linear condition, localization
- Symmetric hyperbolic energy \leftrightarrow Summation by parts in finite differences
- Strongly hyperbolic pseudo-energy \leftrightarrow Pseudo-spectral methods

Applications:

- ADM-BSSN first-second order systems [Frittelli-R, Sarbach-Calabrese-Pullin-Tiglio, Kreiss-Ortiz, Nagy-Ortiz-R]
- Constraint propagation. [Hyperbolicity properties of subsidiary systems of constraints.]

ADM equations I

$$G_{ab} = 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} \mathcal{L}_n h_{ab} = -2k_{ab}, \\ \mathcal{L}_n k_{ab} = {}^{(3)}R_{ab} - 2k_a{}^c k_{bc} + k_{ab} k_c{}^c - \frac{D_a D_b N}{N}, \\ {}^{(3)}R + (k_c{}^c)^2 - k_{ab} k^{ab} = 0, \\ D^b k_{ba} - D_a k = 0, \end{array} \right.$$

ADM equations II

$$\mathcal{L}_{(t-\beta)} h_{ij} = -2Nk_{ij}$$

$$\mathcal{L}_{(t-\beta)} k_{ij} = \frac{N}{2} h^{kl} [-\partial_k \partial_l h_{ij} - \partial_i \partial_j h_{kl} + 2\partial_k \partial_{(i} h_{j)l}] + B_{ij}$$

where

$$B_{ij} := N [\gamma_{ikl} \gamma_j^{kl} - \gamma_{ij}^k \gamma_{kl}^l - 2k_i^l k_{jl} + k_{ij} k_l^l - A_{ij}],$$

$$\gamma_{ij}^k := \frac{1}{2} h^{kl} (2\partial_{(i} h_{j)k} - \partial_k h_{ij}),$$

$$A_{ij} := a_i a_j - \gamma_{ij}^k a_k - 2\gamma_{ikl} \gamma_j^{(kl)} + \partial_i [(\partial_j N)/N],$$

$$a_i := (\partial_i N)/N.$$

ADM equations III

Hyperbolicity analysis: 1) consider only the principal part, 2) freeze coefficients, 3) substitute all derivatives by Fourier transforms ($\partial_k h_{ij} \rightarrow i\omega_k \hat{h}_{ik}$), and 4) define $\hat{\ell}_{ij} = i\omega \hat{h}_{ij}$. [Kreiss, Ortiz][Taylor]

The associated first order system is then

$$\begin{aligned}\partial_t \hat{\ell}_{ij} &\hat{=} i\omega \left[-2N \hat{k}_{ij} + \tilde{\omega}_k \beta^k \hat{\ell}_{ij} \right], \\ \partial_t \hat{k}_{ij} &\hat{=} i\omega \left[-\frac{N}{2} \left(\hat{\ell}_{ij} + \tilde{\omega}_i \tilde{\omega}_j h^{kl} \hat{\ell}_{kl} - 2\tilde{\omega}^k \tilde{\omega}_{(i} \hat{\ell}_{j)k} \right) + \tilde{\omega}_k \beta^k \hat{k}_{ij} \right]\end{aligned}$$

with $\tilde{\omega}_i = \omega_i / \omega$.

Result:

- ADM equations are only weakly hyperbolic (3 eigenvectors missing).
- Comments on weakly hyperbolicity

ADM equations IV

$$N = h^b Q \quad (h = \text{determinant of } h_{ij})$$

The associated first order system is then

$$\partial_t \hat{\ell}_{ij} \hat{=} i\omega \left[-2N \hat{k}_{ij} + \tilde{\omega}_k \beta^k \hat{\ell}_{ij} \right],$$

$$\partial_t \hat{k}_{ij} \hat{=} i\omega \left[-\frac{N}{2} \left(\hat{\ell}_{ij} + (1+b) \tilde{\omega}_i \tilde{\omega}_j h^{kl} \hat{\ell}_{kl} - 2\tilde{\omega}^k \tilde{\omega}_{(i} \hat{\ell}_{j)k} \right) + \tilde{\omega}_k \beta^k \hat{k}_{ij} \right]$$

Result:

- Modified ADM equations for $b > 0$ still weakly hyperbolic (2 eigenvectors missing).
- Adding Hamiltonian constraint does not change hyperbolicity, but does change characteristics.

BSSN equations I

$$f^k = h^{ij} \gamma_{ij}{}^k + dh^{kl} \gamma_{lm}{}^m = h^{kl} (h^{ij} \partial_i h_{jl} + \partial_l \ln h) \quad d = 0$$

$$\mathcal{L}_{(t-\beta)} h_{ij} = -2N k_{ij}$$

$$\mathcal{L}_{(t-\beta)} k_{ij} = \frac{N}{2} h^{kl} [-\partial_k \partial_l h_{ij} - b \partial_i \partial_j h_{kl}] + N \partial_{(i} f_{j)} + \mathcal{B}_{ij}$$

$$\mathcal{L}_{(t-\beta)} f_i = N [-(2 - c) D^k k_{ki} + (1 - c) D_i k_k{}^k] + \mathcal{C}_i$$

BSSN equations II

Hyperbolicity Analysis:

$$\begin{aligned}\partial_t \hat{\ell}_{ij} &\hat{=} i\omega \left[-2\alpha \hat{k}_{ij} + \tilde{\omega}_k \beta^k \hat{\ell}_{ij} \right] \\ \partial_t \hat{k}_{ij} &\hat{=} i\omega \left[\frac{\alpha}{2} \left(-\hat{\ell}_{ij} - b \tilde{\omega}_i \tilde{\omega}_j h^{kl} \hat{\ell}_{kl} + 2\tilde{\omega}_{(i} \hat{f}_{j)} \right) + \tilde{\omega}_k \beta^k \hat{k}_{ij} \right] \\ \partial_t \hat{f}_i &\hat{=} i\omega \left[\alpha \left((-2 + c) \hat{k}_{ik} \tilde{\omega}^k + (1 - c) \tilde{\omega}_i h^{kl} \hat{k}_{kl} \right) + \tilde{\omega}_k \beta^k \hat{f}_i \right]\end{aligned}$$

Result: [Nagy-Ortiz-R]

- Modified BSSN equations for $b > 0$ $c > 0$ strongly hyperbolic.
- Eigenvalues: $(0, \pm 1, \pm \sqrt{b}, \pm \sqrt{c/2})$

Constraint Propagation

Evolution System:

$$\partial_t u^\alpha = A(u, t, x)^{\alpha a}{}_\beta \partial_a u^\beta + B(u, t, x)^\alpha,$$

Constraints:

$$C^A = K(u, t, x)^{Aa}{}_\beta \partial_a u^\beta + G(u, t, x)^A,$$

Integrability condition (subsidiary system):

$$\partial_t C^A = S(u, t, x)^{Aa}{}_B \partial_a C^B + R(u, \partial u, t, x)^A{}_B C^B,$$

- Want to study what can we say about the properties of the subsidiary system from what we know from the evolution system.

Constraint Propagation II

- Problem: In general $S(u, t, x)^{Aa}_B$ is not unique if the constraint themselves satisfy certain identities.

For instance, if there is an $L_A(\omega)$ such that:

$$L_A(\omega) K^{An}{}_{\alpha} \omega_n = 0$$

we could add to $S(u, t, x)^{Aa}_B$

$$M^{Aa} L_B$$

- With this addition there are easy examples where one can get any sort of badly posed systems!

Constraint Propagation III

- Assume: For any ω_i , $K_{\alpha}^{An} \omega_n$ is surjective.
- In general this is not satisfied, but in examples of interest one finds subset of constraints which do satisfy it. [Maxwell, EC].

Constraint Propagation IV

Integrability condition implies:

$$K^{A(a}{}_{\alpha} A^{|\alpha|b)}{}_{\beta} - S^{A(a}{}_{B} K^{|B|b)}{}_{\beta} = 0$$

- **Lemma 1:** Given any fixed non-vanishing co-vector ω_a . If (σ, u^{α}) is an eigenvalue-eigenvector pair of $A^{\alpha a}{}_{\beta} \omega_a$ then $(\sigma, v^A = K^{Aa}{}_{\alpha} \omega_a u^{\alpha})$, if v^A is non-vanishing, is an eigenvalue-eigenvector pair of $S^{Aa}{}_{B} \omega_a$.

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- **Corollary 1:** If the evolution system is strongly hyperbolic then so is the subsidiary system. [It does not work symmetric \rightarrow symmetric].
- **Corollary 2:** The characteristics of the subsidiary system are a subset of the characteristics of the evolution system. The domain of dependence of the subsidiary system is at least as large as the domain of dependence of the evolution system.