Radiative-type boundary conditions for Einstein’s equations

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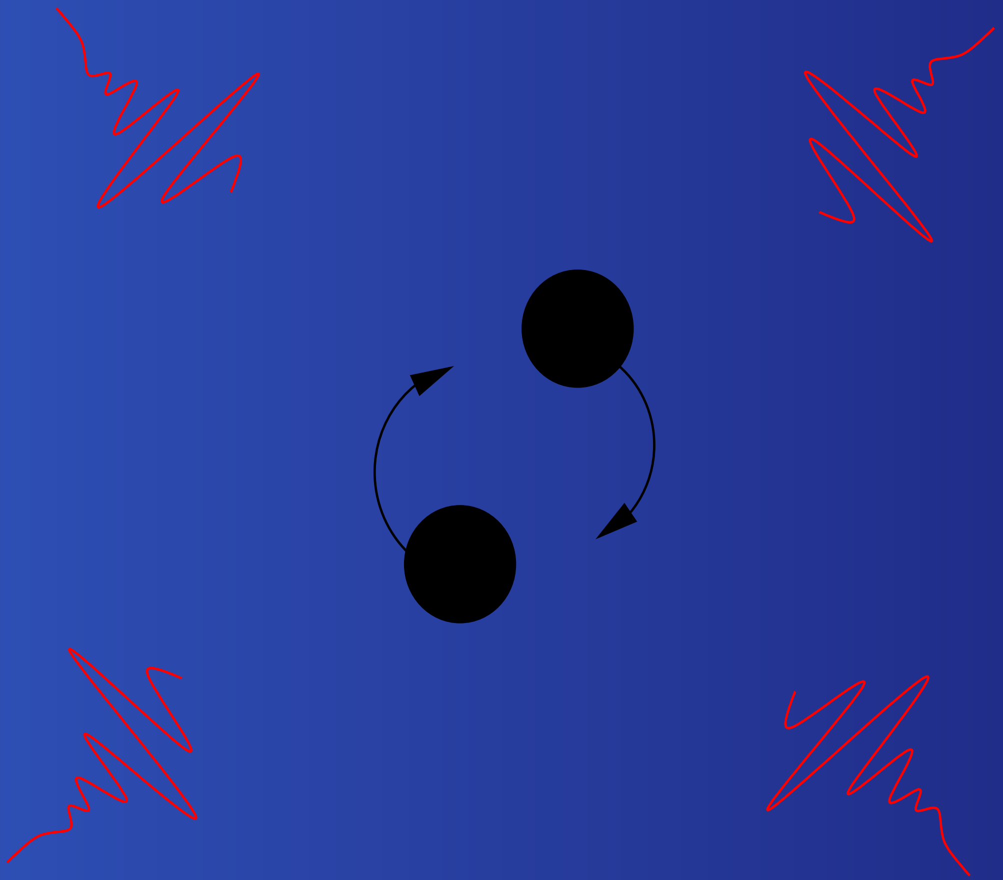
Outline

- Absorbing outer boundaries
- The freezing Ψ_0 boundary condition
- Towards well posed intial-boundary value formulations
- The Bianchi equations and linearized gravity
- Backscattering
- Conclusions

Absorbing outer boundaries

Ideal outer boundary is transparent



Absorbing outer boundaries

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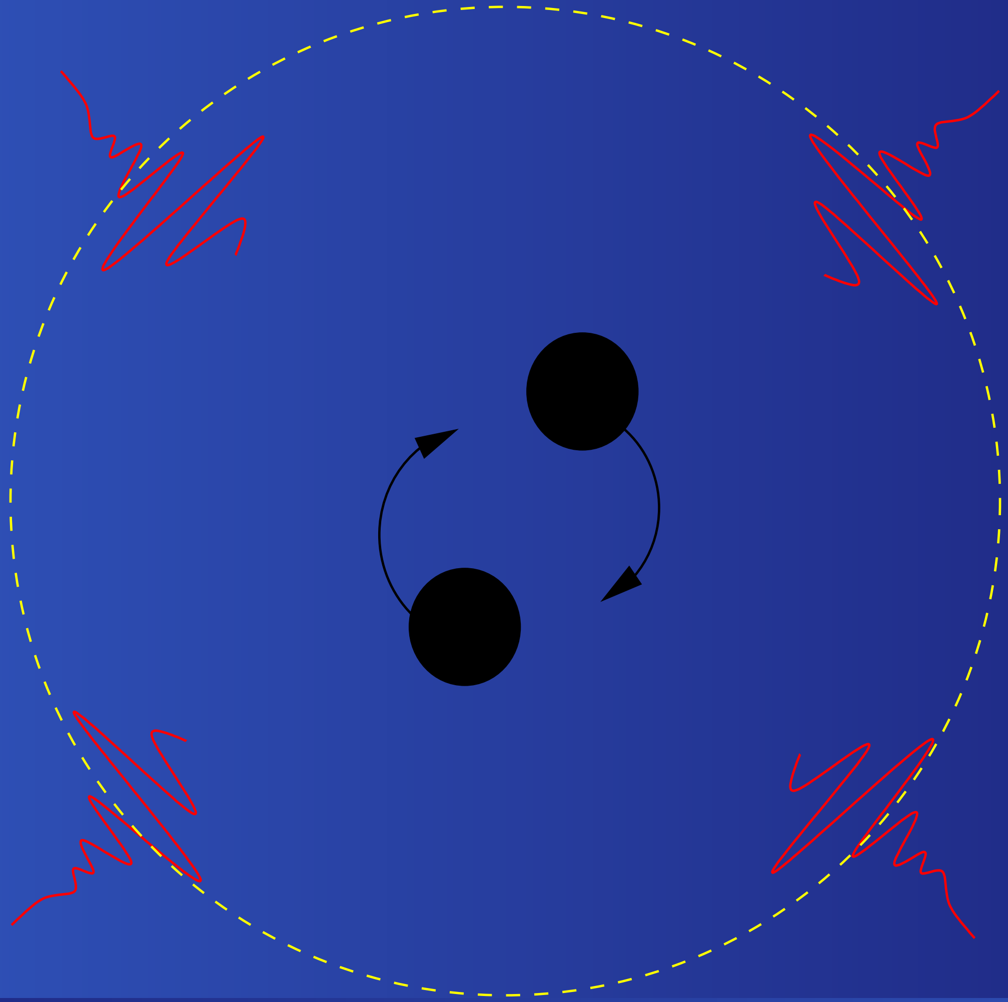

Absorbing outer boundaries

Replace unbounded domain by a bounded (compact) domain with artificial outer boundary.

Ideally, the artificial boundary is completely transparent to the physical problem on the unbounded domain.

Realistically, shoot for boundary conditions which are absorbing in the following sense:

- They form a well-posed initial boundary value problem (IBVP).
- They ensure that very little *spurious* reflections of gravitational radiation occur from the outer boundary.
- Error with respect to solution to global problem can be estimated.

Absorbing outer boundaries

Examples:

- 1D wave equation

$$\left(\partial_t^2 - \partial_x^2\right) u(t,x) = 0, \qquad t > 0,\ x \in [-1,1].$$

General solution is superposition of left- and right-moving solution,

$$u(t,x) = f_{\nwarrow}(x+t) + f_{\nearrow}(x-t),$$

so the boundary conditions

$$(\partial_t - \partial_x)u(t,-1) = 0, \qquad (\partial_t + \partial_x)u(t,+1) = 0,$$

are perfectly absorbing.

Absorbing outer boundaries

Examples:

- 3D wave equation (much more difficult because of modes propagating tangentially to the boundary!)
  Spherical harmonic decomposition

$$u(t,r,\vartheta,\varphi) = \frac{1}{r}\sum_{\ell=0}^{\infty}\sum_{|m|\leq\ell} u_{\ell m}(t,r) Y^{\ell m}(\vartheta,\varphi)$$

yields

$$\left(\partial_t^2 - \partial_r^2 + \frac{\ell(\ell+1)}{r^2}\right) u_{\ell m}(t,r) = 0, \qquad t > 0,\ r \in (0,R).$$

Solutions can be generated from the 1D solutions by applying suitable differential operators to them.

Absorbing outer boundaries

Define the operators $a_\ell \equiv \partial_r + \frac{\ell}{r}$, $a_\ell^\dagger \equiv -\partial_r + \frac{\ell}{r}$.
They satisfy the operator identities

$$a_{\ell+1} a_{\ell+1}^\dagger = a_\ell^\dagger a_\ell = -\partial_r^2 + \frac{\ell(\ell+1)}{r^2}\; .$$

As a consequence, for each $\ell = 1, 2, 3...$,

$$\begin{aligned}
\left[\partial_t^2 - \partial_r^2 + \frac{\ell(\ell+1)}{r^2}\right] a_\ell^\dagger a_{\ell-1}^\dagger ... a_1^\dagger &= \left[\partial_t^2 + a_\ell^\dagger a_\ell\right] a_\ell^\dagger a_{\ell-1}^\dagger ... a_1^\dagger \\
&= a_\ell^\dagger \left[\partial_t^2 + a_{\ell-1}^\dagger a_{\ell-1}\right] a_{\ell-1}^\dagger ... a_1^\dagger \\
&= a_\ell^\dagger a_{\ell-1}^\dagger ... a_1^\dagger \left[\partial_t^2 - \partial_r^2\right].
\end{aligned}$$

Absorbing outer boundaries

Therefore, we have the explicit in- and outgoing solutions

$$\begin{aligned}\phi_{\nwarrow,\ell}(t,r) &= a_\ell^\dagger a_{\ell-1}^\dagger ... a_1^\dagger V_\ell(r+t),\\ \phi_{\nearrow,\ell}(t,r) &= a_\ell^\dagger a_{\ell-1}^\dagger ... a_1^\dagger U_\ell(r-t).\end{aligned}$$

- Lemma

  Let $b_- = r^2(\partial_t + \partial_r)$. *Then,* $b_-^{\ell+1}\phi_{\nearrow,\ell}(t,r) = 0$ *for all* $\ell = 0, 1, 2, ...$.

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- Therefore, given $L \in \{1, 2, 3, \ldots\}$, the boundary condition

$$\mathcal{B}_L : \quad b_-^{L+1}(ru)(t,r,\vartheta,\varphi)\big|_{r=R} = 0$$

leaves the outgoing solutions with $\ell \leq L$ unaltered.

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  leaves the outgoing solutions with $\ell \leq L$ unaltered.
- Furthermore, one can show that each boundary condition $\mathcal{B}_L$ yields a well posed problem (and solutions are unique).

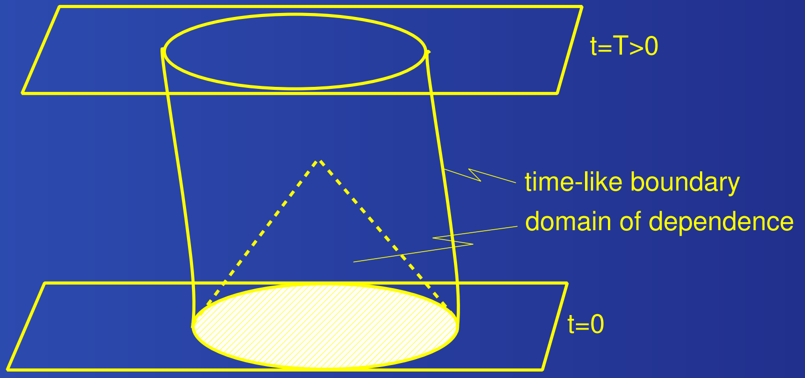
Absorbing outer boundaries

- Therefore, $\mathcal{B}_L$ is perfectly absorbing for waves with $\ell \leq L$.
- Notice that in most scenarios, the major contribution stems from the lower few ℓ's.
- Hierarchy of *local* boundary conditions with increasing order of accuracy.
  Bayliss and Turkel, Comm. Pure and Appl. Math., **33**, 707-725 (1980)

Absorbing outer boundaries

In GR: A Challenging Problem!

- Do not know *a priori* the future geometry of the outer boundary
- Presence of constraints which propagate *across* the boundary
- It is difficult to define what is meant by outgoing and ingoing radiation because of nonlinear nature of theory



Absorbing outer boundaries

In GR: A Challenging Problem!

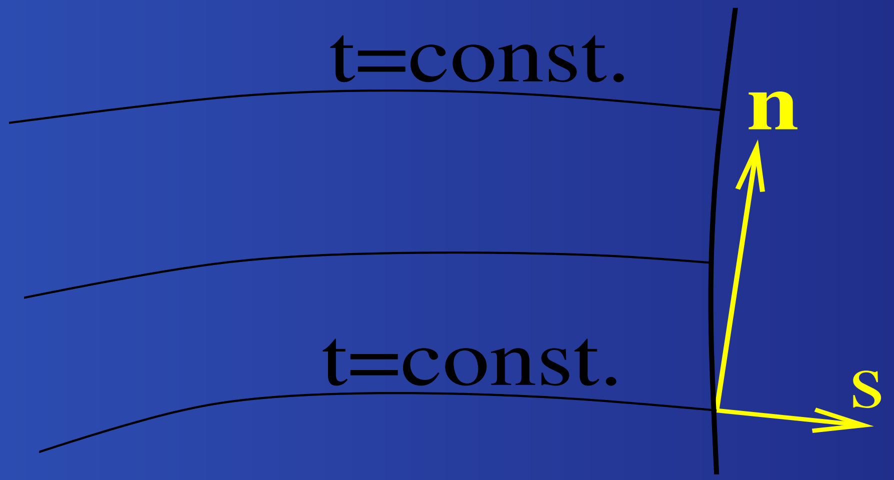
- Friedrich & Nagy, 1999: Well posed IBVP in terms of a first order symmetric hyperbolic tetrad-based system involving the Weyl tensor as a dynamical variable.
- Novak & Bonazzola, 2004: Numerical implementation of the $L = 2$ Bayliss-Turkel boundary condition for the wave equation.
- Lau, 2004, 2005: Fast converging series expansion of *exact nonlocal* boundary conditions for the Regge-Wheeler and Zerilli equations.
- Buchman & S, 2006: Hierarchy $\mathcal{B}_L$ of *local* boundary conditions which are exact for linearized gravitational radiation with angular momentum $\ell \leq L$.

The freezing Ψ_0 BC

Construct tetrad at the boundary surface:

n: unit future-directed normal to the time slices

s: unit outward normal to the cross sections (tangential to $t = const$).



The freezing Ψ_0 BC

Construct tetrad at the boundary surface:

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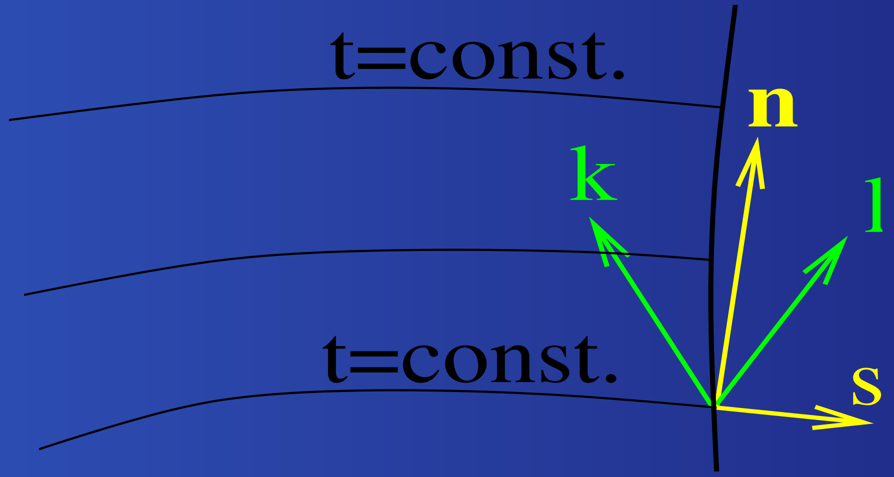
s: unit outward normal to the cross sections (tangential to $t = const$).

Two null vectors, $k = \frac{1}{\sqrt{2}}(n - s), \qquad l = \frac{1}{\sqrt{2}}(n + s).$



The freezing Ψ_0 BC

- Define the Weyl scalars

$$\begin{aligned} \Psi_0 &= C_{abcd} l^a m^b l^c m^d \qquad &\text{(incoming radiation at null infinity)},\\ \Psi_4 &= C_{abcd} k^a \bar{m}^b k^c \bar{m}^d \qquad &\text{(outcoming radiation at null infinity)}, \end{aligned}$$

  where C_{abcd} is the Weyl tensor and m^a is a complex null vector orthogonal to l^a and k^a which satisfies $m^a \bar{m}_a = 1$.

- Peeling theorem (Penrose, 1965):
  $\Psi_j = O(r^{j-5})$, $j = 0, 1, 2, 3, 4$, along outgoing null geodesics.
  So one possibility is to set $\Psi_0 = 0$ at the boundary.

- For compatibility with the initial data, set

$$\partial_t \Psi_0 |_{\partial\Omega} = 0. \qquad \text{(freezing } \Psi_0 \text{ boundary condition).}$$

The freezing Ψ_0 BC

- Ψ_0 is gauge-invariant quantity in linearized gravity.
- Ψ_0 is gauge-invariant for linear perturbations of a type D metric (for the correct choice of background tetrad).

The freezing Ψ_0 BC

- Friedrich & Nagy, 1999: Well posed IBVP in terms of a first order symmetric hyperbolic tetrad-based system involving the Weyl tensor as a dynamical variable.
- S & Tiglio; Kidder, Lindblom, Pfeiffer, Scheel, Teukolsky, 2005: Numerically implemented in the Einstein-Christoffel formulation (some weak instabilities observed)
- Kidder, Lindblom, Pfeiffer, Rinne, Scheel, Teukolsky, 2006: Numerically implemented in a first order reduction of the harmonic system.
- Nagy & S, 2006: Formulation based on the standard (York) $3+1$ decomposition with elliptic gauge conditions; complete well posedness proof in the linearized case using semi-group methods.

Well posedness of the IBVP

Einstein's vacuum field equations with (generalized) harmonic coordinates.

M. Ruíz & S, in progress

$$g^{ab}\left(\Gamma^{c}{}_{ab} - \mathring{\Gamma}^{c}{}_{ab}\right) = H^{c},$$

where $\mathring{g}_{ab}$ is a fixed background metric with corresponding Christoffel symbols $\mathring{\Gamma}^{c}{}_{ab}$ and H^{c} a given vector field.

Einstein's vacuum equations:

$$\mathring{g}^{cd}\mathring{\nabla}_{c}\mathring{\nabla}_{d}g_{ab} = S_{ab}\ ,$$

where S_{ab} depends (quadratically) on first derivatives of g_{ab}.

Ten wave equations, so we need ten boundary conditions.

Well posedness of the IBVP

Constraint-preserving boundary conditions (4 conditions):

$$l^a \nabla_a \mathcal{C}_b \big|_{\partial\Omega} = 0, \qquad \mathcal{C}^c \equiv g^{ab}\left(\Gamma^c{}_{ab} - \mathring{\Gamma}^c{}_{ab}\right) - H^c .$$

Radiation controlling boundary condition, with q some data (2 conditions):

$$\Psi_0\big|_{\partial\Omega} = C_{abcd} l^a m^b l^c m^d \Big|_{\partial\Omega} = q .$$

Other 4 conditions, with p, π, q_1 some data:

$$\begin{array}{rcl} l^a l^b l^c l^d \mathring{\nabla}_a \mathring{\nabla}_b g_{cd}\Big|_{\partial\Omega} & = & p \ , \\ l^a l^b l^c k^d \mathring{\nabla}_a \mathring{\nabla}_b g_{cd}\Big|_{\partial\Omega} & = & \pi \ , \\ l^a l^b l^c m^d \mathring{\nabla}_a \mathring{\nabla}_b g_{cd}\Big|_{\partial\Omega} & = & q_1 \ . \end{array}$$

Well posedness of the IBVP

Frozen coefficient approximation: Consider small amplitude, high frequency perturbations of a smooth background solution.

- System reduces to a linear, constant coefficient problem on a half space $\Omega = \{x > 0\}$ domain.
- By performing a suitable coordinate transformation which leaves Ω invariant, one can achieve that at any point on the boundary, $\mathring{g} = -dt^2 + (dx + \beta\, dt)^2 + dy^2 + dz^2$. ($\beta$: “shift”).
- We obtain a system of ten constant coefficient wave problems,

$$\begin{aligned} \left[-\partial_t^2 + 2\beta\partial_t\partial_x + (1-\beta^2)\partial_x^2 + \partial_y^2 + \partial_z^2\right] h_{ab} &= S_{ab} \qquad \text{on } \Omega, \\ \left[\partial_t - (1+\beta)\partial_x\right]^2 h_{ab}\Big|_{\partial\Omega} &= q_{ab}, \end{aligned}$$

where q_{ab} depends on *tangential* derivatives of h_{ab}.

Well posedness of the IBVP

- Difficulty: q_{ab} depends on first order *tangential* derivatives of h_{ab}. Standard energy estimate techniques do not work.
- One gets an energy estimate using the Kreiss theory and can show that (in the frozen coefficient approximation) the problem is well posed.
- Generalization to the nonlinear case possible via the theory of pseudo-differential operators.

The Bianchi equations

Try to understand the problem completely in the weak field regime.
L. Buchman and OS, CQG, **23**, 6709–6744 (2006)

- Weak field gravity:

  $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ ,$$

  where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is a small ($|h_{\mu\nu}| \ll 1$) perturbation. Neglect quadratic and higher order terms in $h_{\mu\nu}$.

- Let the domain be a ball B_R of radius R.

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  OK if boundary is far enough in the wave zone.
- Let the domain be a ball B_R of radius R.
  Not absolutely necessary. In any case, some codes out there can handle spherical boundaries (LSU-FaMAF mutlipatch, Caltech-Cornell pseudo-spectral,...).

The Bianchi equations

- Weak gravitational waves are conveniently described by the vacuum Bianchi equations,

  $$\nabla^a C_{abcd} = 0,$$

  where C_{abcd} is the linearized Weyl tensor.
- Linearized Weyl tensor is *invariant* with respect to infinitesimal coordinate transformations, $h_{\mu\nu} \mapsto h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$.
- $3+1$ decomposition yields a symmetric hyperbolic first order system similar to Maxwell's equations.
- Expand the linearized Weyl tensor in spherical tensor harmonics.
- The Bianchi equations can be coupled to equations for metric and connection variables, giving the full nonlinear Einstein equations.

The Bianchi equations

Result:

- For $\ell = 0$ and $\ell = 1$ the solutions are essentially non-dynamical, and describe (among others) the linearized Kerr solution (where the linearization is in the mass M and the angular momentum J).
- For $\ell \geq 2$, the solutions can be reduced to two *master equations*. (i.e. the linearized Weyl tensor can be reconstructed from the solutions of these two master equations.)

The Bianchi equations

- One master equation describing the evolution of constraint violations,

  $$\left[\frac{1}{c^2}\partial_t^2 - \partial_r^2 + \frac{\ell(\ell+1)}{r^2}\right]\pi(t,r) = 0,$$

  where $c = 1/2$ (half the speed of light).
- One master equation describing the evolution of gravitational radiation,

  $$\left[\partial_t^2 - \partial_r^2 + \frac{\ell(\ell+1)}{r^2}\right]\psi_2(t,r) = S(t,r),$$

  where the source term $S(t,r)$ depends on π.
- Notice that these equations are invariant with respect to time reversal. Therefore, in- and outgoing solutions are related to each other by $t \mapsto -t$. Teukolsky formalism: more complicated!

Designing boundary conditions

- First try: Based on the symmetric hyperbolic structure of the evolution equations, freeze the incoming characteristic fields to their initial values.
- This yields a well posed IBVP by standard theorems. On the other hand, the solutions obtained are *incompatible with the constraints* $\pi = 0$: It is possible to construct exact solutions which satisfy the constraints at $t = 0$ but violate them at later times.
- Next step: One can construct so-called *constraint-preserving boundary conditions* which guarantee that a solution which satisfies $\pi = 0$, $\partial_t \pi = 0$ at $t = 0$ automatically satisfies the constraints at each time.

Designing boundary conditions

- If the constraints are satisfied, the linearized Weyl tensor is entirely determined by the solution ψ_2 of the master equation.
- It is convenient to group the ten components of the linearized Weyl tensor into five complex scalars Ψ_0, Ψ_1, Ψ_2, Ψ_3 and Ψ_4 which are defined in terms of the null tetrad $l = (\partial_t + \partial_r)/\sqrt{2}$, $k = (\partial_t - \partial_r)/\sqrt{2}$, m, $\bar{m}$. For the exact outgoing solutions constructed in the previous section one can show that along outgoing null geodesics ($t - r = const.$)

$$\Psi_j = O(r^{j-5}), \quad j = 0, 1, 2, 3, 4.$$

Peeling theorem, Penrose, 1965.

- Therefore, as a first approximation for handling the physical fields, one can set

$$\Psi_0\big|_{r=R} = 0.$$

Designing boundary conditions

Reflection Coefficients for the BC with $\Psi_0 = 0$

- The exact outgoing solutions do not satisfy this boundary condition exactly: Ψ_0 falls off as $1/r^5$ along the outgoing null radial geodesics.
- A solution to the IBVP corresponding to $\Psi_0 = 0$ consists of a superposition of an out- and an ingoing wave.
- In order to quantify the amount of reflections, we make the monochromatic ansatz

$$\psi_2(t,r) = a_\ell^\dagger a_{\ell-1}^\dagger ... a_1^\dagger \left(e^{ik(r-t)} + \gamma e^{-ik(r+t)} \right),$$

where γ is an *amplitude reflection coefficient*.

Designing boundary conditions

Reflection Coefficients dor the BC with $\Psi_0 = 0$

- Result:

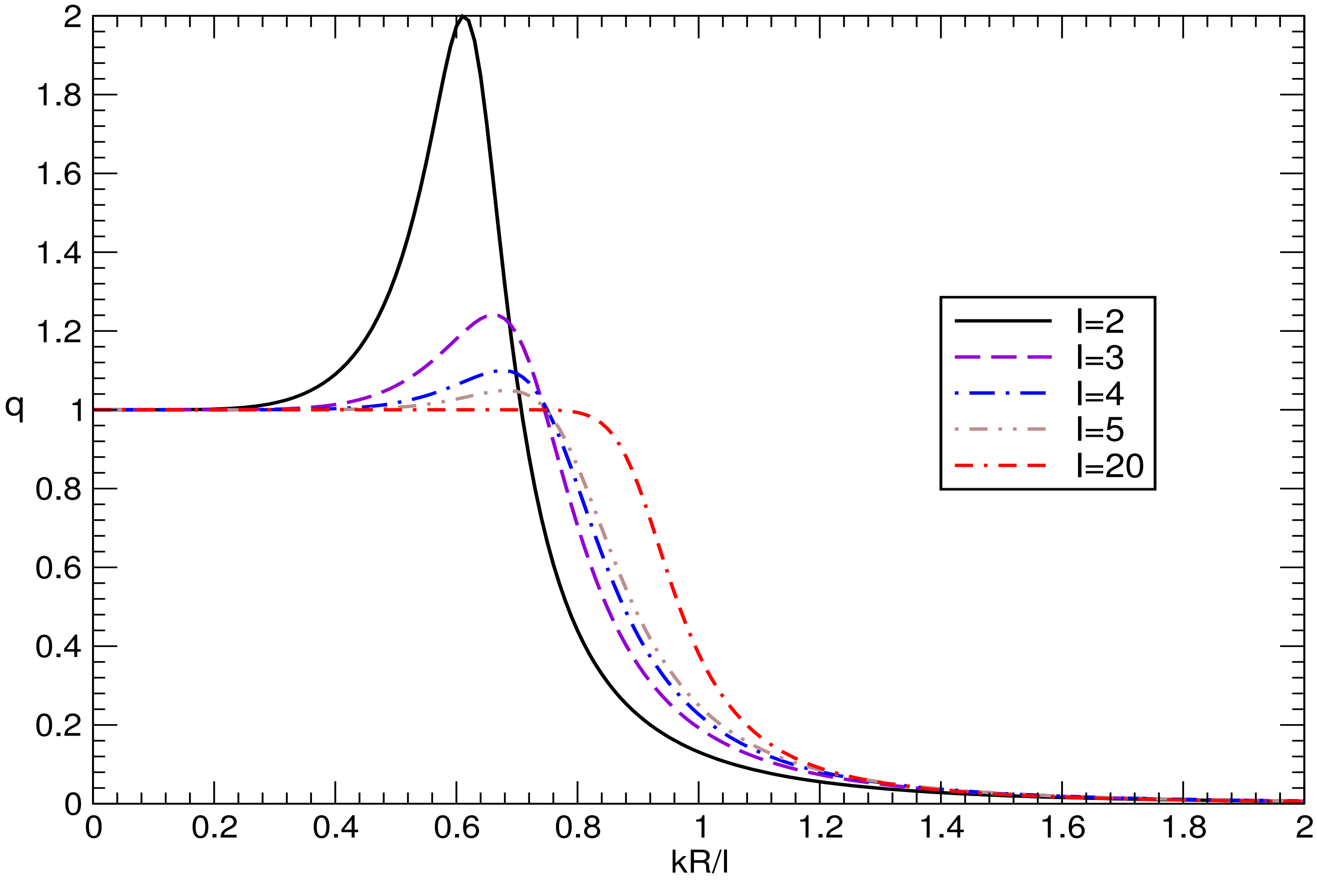
$$q \equiv |\gamma| = \left| \frac{p_{\ell,-2}(-ikR)}{p_{\ell,2}(ikR)} \right|$$

where the polynomials $p_{\ell,m}(z)$, $|m| \leq \ell$, are given by

$$p_{\ell,m}(z) = \sum_{j=0}^{\ell+m} \frac{(\ell+m)!\,(2\ell-j)!}{(\ell+m-j)!\,j!}\,(2z)^j .$$

- $|\gamma|$ is of order unity if $kR < \ell$, and decays as $(kR)^{-4}$ for large kR/ℓ.

Reflection Coefficient vs. kR/ℓ



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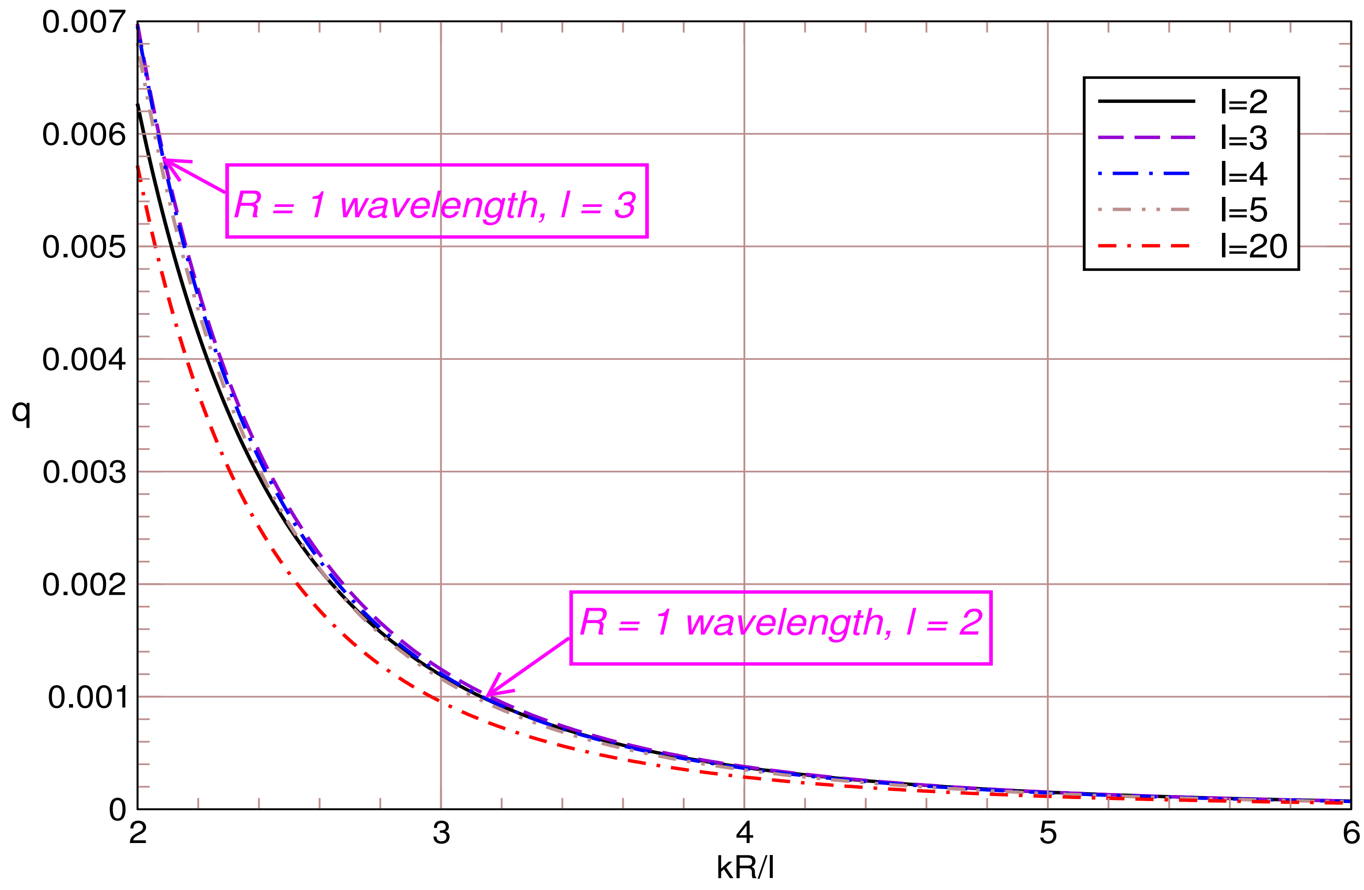
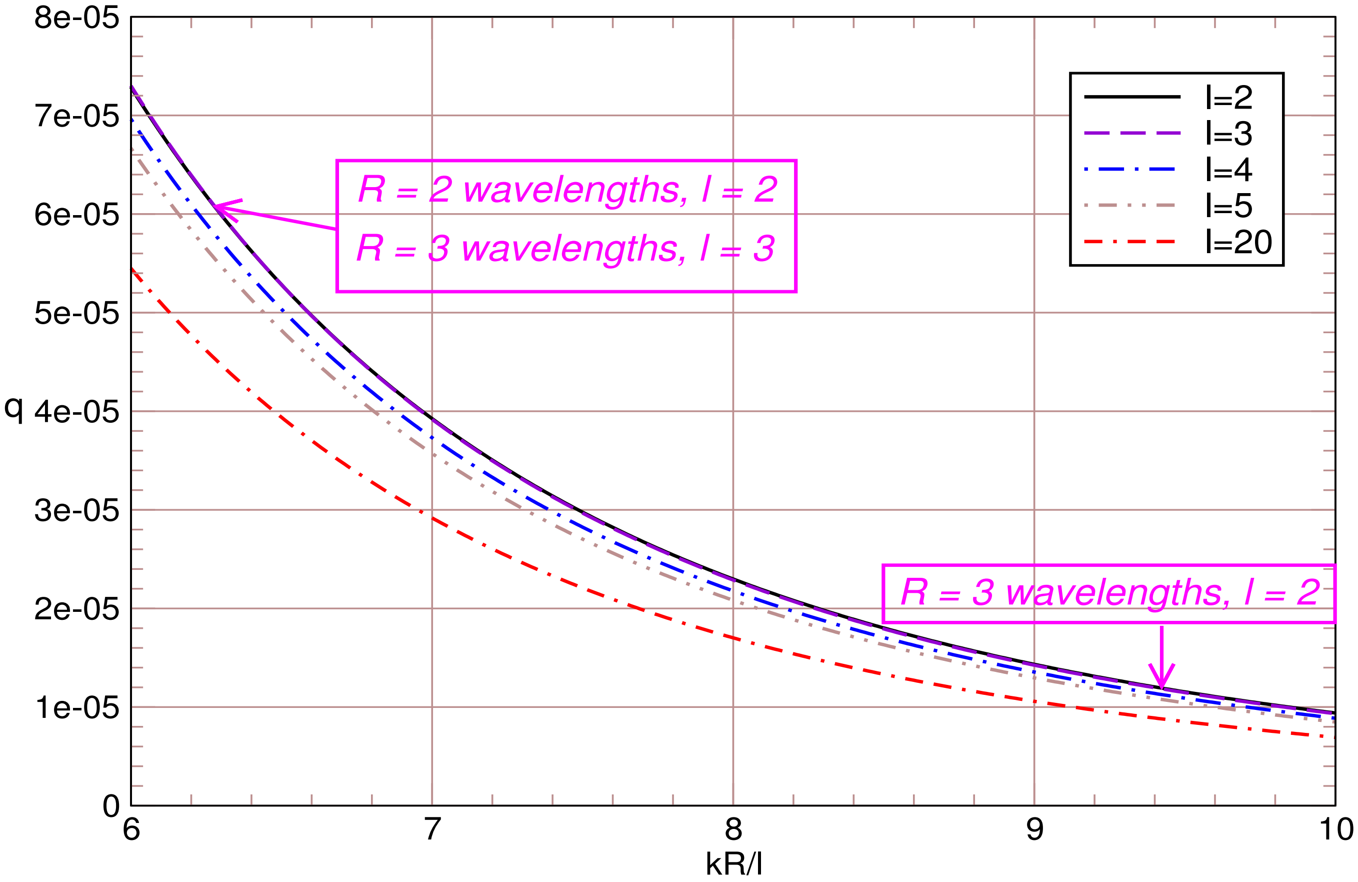


R = 1 wavelength, l = 3

R = 1 wavelength, l = 2

Reflection Coefficient vs. kR/ℓ



R = 2 wavelengths, l = 2

R = 3 wavelengths, l = 3

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Designing boundary conditions

Perfectly absorbing b. c. for linearized waves

- Relation between Ψ_0 and ψ_2:

$$r^5\Psi_0 \sim (b_-)^2\psi_2, \qquad b_- = r^2(\partial_t + \partial_r).$$

- Therefore, setting Ψ_0 to zero at the outer boundary corresponds to the Bayliss-Turkel boundary condition on ψ_2 for $L = 1$.
- Generalization to higher ℓ's:

$$\mathcal{B}_L : \quad (b_-)^{L-1}(r^5\Psi_0) = 0\Big|_{r=R}.$$

- Hierarchy $\mathcal{B}_L$ of improved local BC which are perfectly absorbing for all gravitational waves with angular momentum number $\ell \leq L$, for each $L \geq 2$.

Designing boundary conditions

Perfectly absorbing b. c. for linearized waves

- In many practical situations, expect the few lower multipoles to dominate, so an implementation of this boundary condition for $L = 2, 3$ or 4 should suppress much of the spurious reflection.
- For $L = 2$:

$$(\partial_t + \partial_r)\partial_t(r^5\Psi_0) = 0.$$

- Reflection coefficients for $\ell > L$:

$$|\gamma| = \left|\frac{p_{\ell,-L+1}(-ikR)}{p_{\ell,L+1}(ikR)}\right|$$

decays as $(kR)^{-2(L+1)}$ for large kR.

Backscattering

- Outer boundary lies in the weak field regime => can describe the background near the outer boundary by the Schwarzschild metric with mass M where M represents the total mass of the system.

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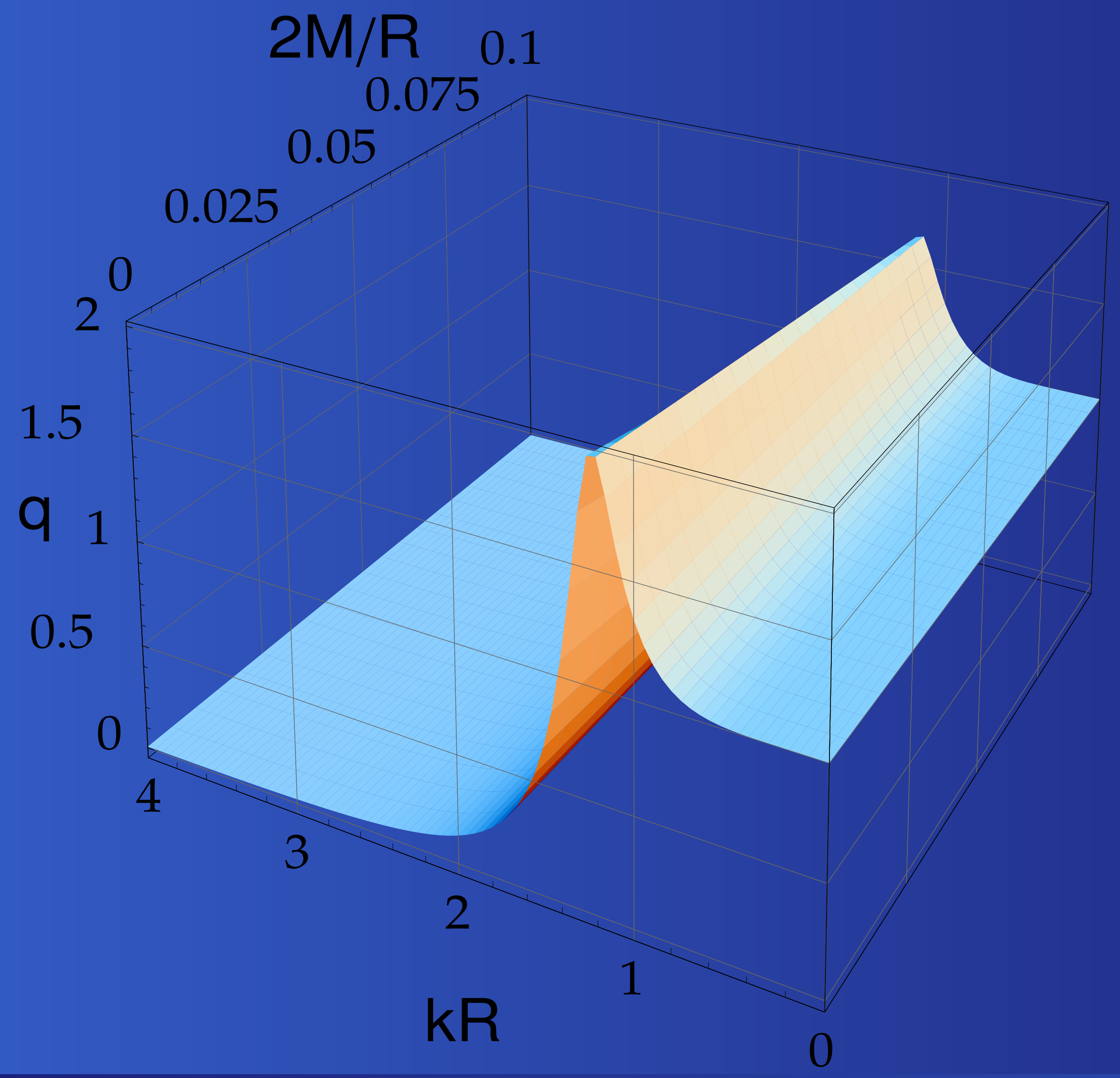
- Result ($\Psi_0 = 0$): For $2M/R \ll 1$, the corrected $\ell = 2$ reflection coefficient depends only weakly on $2M/R$.

Reflection Coefficient for $\ell = 2$



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- Result ($\Psi_0 = 0$): For $2M/R \ll 1$, the corrected $\ell = 2$ reflection coefficient depends only weakly on $2M/R$.
- Result ($L = 2$ improved BC): Reflection coefficient is smaller than the previous one by a factor of M/R for $kR > 1.05$.

Backscattering

- Only analyzed odd-parity sector so far. In this case, the master equation for $Im(\Psi_2)$ is given by the Regge-Wheeler equation

  $$\left[\partial_t^2 - \partial_{r_*}^2 + \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right)\right]\psi_2(t,r) = 0,$$

  where r_*: tortoise coordinate.

- Expansion in $2M/R$: Define new coordinates $\tau = t + r - r_*$, $\rho = r$ and expand

  $$\psi_2(\tau,\rho) = a_\ell(\rho)^\dagger a_{\ell-1}(\rho)^\dagger ... a_1(\rho)^\dagger U(\rho - \tau) + \sum_{k=1}^{\infty}\left(\frac{2M}{R}\right)^k g_k(\tau,\rho),$$

  => $\left[\partial_\tau^2 - \partial_\rho^2 + \frac{\ell(\ell+1)}{\rho^2}\right] g_1(t,\rho) = S[U]$.

Backscattering

- First order correction:

$$g_1(\tau,\rho) = \frac{3R}{4\rho^2}U'(\rho-\tau) + \frac{R}{4}\int\limits_{\rho-\tau}^{\infty} K_2(\tau,\rho,x)U(x)dx,$$

where the integral kernel K_2 is given by

$$K_2(\tau,\rho,x) = \frac{3}{2\rho^4}\left[w^{-4} + 2w^{-3} + 2w^{-2}\right]_{w=\frac{\tau+\rho+x}{2\rho}}, \qquad x > \rho - \tau.$$

- Corresponding ingoing solution by reversing the sign of t.
- $M = 0$: Huygen's principle: “flash” at $(\tau,\rho) = (0,0)$ propagates along null rays $t = \pm r$.
- $M > 0$: “Flash” at $(\tau,\rho) = (0,0)$ yields nonvanishing perturbation in the whole future cone $|\rho| \leq \tau$.

Backscattering

Reflection coefficient for improved BC with $L = 2$

- $|\gamma| = \frac{2M}{R} E(kR) + O(2M/R)^2$ where

$$E(z) = \frac{3}{4z^5}\left[(1 - 15C_7(z))^2 + (z - 15S_7(z))^2\right]^{1/2}.$$

$$C_n(z) = \int\limits_0^\infty \frac{\cos(2zy)}{(1+y)^n}\,dy, \qquad S_n(z) = \int\limits_0^\infty \frac{\sin(2zy)}{(1+y)^n}\,dy, \qquad n = 1, 2, 3, \dots$$

- For large z, $E(z)$ decays like z^{-4}.
- However, due to the small factor $2M/R$, we get a much smaller reflection coefficient than before.

Conclusions

- Use of freezing Ψ_0 boundary condition, together with constraint-preserving BC and BC controlling part of the gauge should serve as good first candidate for an absorbing BC.
- Reflection coefficients provide a way to estimate the error in the energy flux due to spurious reflections.
- Constructed improved BC which are exact for linearized gravitational radiation with angular momentum ℓ up to a given L.
- Well posedness proof on the way for different formulations (harmonic system, BSSN).
- This work should be useful for binary black hole simulations.
  1. New boundary conditions should improve accuracy.
  2. The new boundary conditions may also be useful to minimize reflections of “junk” radiation present in the initial data.